Application Form

Profile Information

Registration ID

[redacted]

Full Name

[redacted]

Date of Birth

[redacted]

Email Address

[redacted]

Pho

[redacted]

Address

[redacted]

Gender

[redacted]

Ethnicity

[redacted]

Language Preference

[redacted]

Legal Residence Information

Citizenship

[redacted]

State of Legal Residence Mailing Address

[redacted]

Address (line 2)

[redacted]

City

[redacted]

Please identify your state (or U.S. territory) of legal residence

[redacted]

Zip Code

[redacted]

Permanent Phone

[redacted]

For your state of legal residence, select your U.S. Congressional House District

[redacted]

Is your state of legal residence the same as your permanent address? (Usually the address of your parent/legal guardian)

[redacted]

Career Goals/Professional Aspirations

What is the highest degree you plan to obtain?

Ph.D.

In one or two sentences, describe your career goals and professional aspirations (see example below). This statement will be used in publications if you are selected as a scholar or honorable mention.

PhD. in Pure Mathematics. I will one day conduct research in representation theory and commutative algebra and will eventually become a research professor.

What are your career goals and professional aspirations? Indicate which area(s) of mathematics, science or engineering you are considering pursuing in your research career and specify how your current academic program and your overall educational plans will assist you in achieving your career goals and professional aspirations.

I have wanted to become a researcher for as long as I can remember. However, my time in an experimental physics lab and then in a theoretical physics research group taught me something that, deep down, I think I've always known: mathematics is far more natural and beautiful to me than the messy world of empirical science. Two of my personal heroes, Emmy Noether and Paul Dirac, made huge contributions to our understanding of the world not through experimentation, but by logically following sets of assumptions to their mathematical conclusions. I want to become a pure math researcher because I hope to follow in Noether and Dirac's footsteps – to mutually push the boundaries of math and physics. Specifically, I am interested in using tools from representation theory, commutative algebra and algebraic geometry to study problems in cosmology and condensed matter. As an undergraduate, I am preparing for my future career by taking challenging courses, participating in independent projects, and making a conscious effort to develop my communication skills.

Math is perhaps the oldest field of science, so there is much to learn before one can engage in independent research. Graduate courses have exposed me to areas of math not often encountered at the undergraduate level, such as representation theory and commutative algebra, and have provided me with a diverse mathematical background. I hope this will enable me to take my qualifying exams upon entering graduate school, and thereby begin research as soon as possible.

In the meantime, through REUs and independent reading projects, I am engaging in the type of math I hope to study throughout my career. In the past, I completed supervised reading courses in differential manifolds and algebraic number theory but am now focused on topics in mathematical physics. My summer research fell into this category, since the math was both rewarding in its own right and applicable to problems in cosmology. Supervised by my mentor, I read notes on the representations of compact groups as a continuation of this research. I hope to draw upon my background in representation theory to delve into the study of quantum groups during my senior year, with special emphasis on how they arise in the study of non-abelian braiding in quantum Hall systems. These projects, coupled with my double major in physics, will hopefully set me up to become an effective researcher in mathematical physics.

Finally, I make a concerted effort to develop collaboration and communication skills, as these are incredibly important to mathematicians but are not adequately developed over the course of an undergraduate degree. I work as a tutor during the school year, attend conferences whenever possible, and deliver presentations on topics that interest me. In this way I hope to develop the skills to effectively collaborate with scientists and other mathematicians, and ultimately to become the leader of my own research group.

Describe an activity or experience that has been important in helping shape or reinforce your desire to pursue a research career in science, mathematics or engineering.

I attended a local community college during my senior year of high school, and during the spring semester of 2017 I developed an original research project under the supervision of my mentor. I presented the results at the 2017 Salt Lake Community College Science Symposium.

In the months prior to the project, I read a lot about the compounding threat space debris posed to operations in low earth orbit (LEO). Objects in LEO move at such high velocities that particles as small as 0.5 cm along their widest axis can smash through satellites and human-occupied spacecraft. If we could reliably track these objects from the ground, then the viability of LEO operations would be far more certain. I came up with two ideas for novel detection techniques and developed mathematical models to estimate their feasibility. The most promising idea was to identify a cloud of debris by detecting its reflected sunlight.

I eventually concluded that my ideas were largely infeasible, but rather than deterring me, the experience proved incredibly rewarding. The process of identifying a question, developing a model, and providing a previously unknown answer was intensely rewarding. Furthermore, in sharing my conclusions with a wider audience I felt that I had, in some small way, helped others learn something. Since then it has been my goal to lead my own research team.

</br>(Optional question, answering the question below will depend on your personal experience.)</br></br>Goldwater Scholars will be representative of the diverse economic, ethnic and occupational backgrounds of families in the United States. Describe any social and/or economic impacts you have encountered that influenced your education - either positively or negatively - and how you have dealt with them.

The curve connecting a mathematical idea to a fully-fledged proof is anything but simple – it is littered with dead ends and self-intersections. Open-mindedness and persistence in the face of failure are therefore vital to a mathematician. I come from a military family who, every three years, packed up and moved between Europe, the US, and the Middle East. This style of life dramatically broadened my appreciation for the ideas and opinions of others, but more importantly it repeatedly forced me to admit when I was wrong.

However, despite the experiences it provided, my mobile childhood rendered a traditional education impossible. Starting in first grade I was homeschooled, and from seventh grade onwards I taught myself solely through online, self-paced classes. Complete autonomy let me tailor my education around my interests, but I had no teachers to guide me and no opportunities to distinguish myself. I nevertheless took advantage of the flexibility afforded me. I taught myself to build websites and program in C#. I learned basic breadboarding and microcontroller programming, skills I used to build a Transcranial Direct Current Stimulation device. Coursework was a huge focus as well, and by the time I moved to Utah in my senior year, I decided to enroll in community college in lieu of high school. The self-discipline I developed through online classes paid off, and I advanced quickly through all of the physics and math courses the college offered.

Research Projects

Research Project #1
Automorphisms and Faithful Matrix Representations of almost Abelian Lie Groups

Starting Month
06

Starting Year
2019

Ongoing
No

Ending Month
08

Ending Year
2019

Average Hours/Week (Academic Year)
4

Average Hours/Week (Summer)
54

Name of Project Mentor
Zhirayr Avetisyan

Title of Project Mentor
Professor

Description of research, including your involvement in AND contribution to the project.
At UCSB, I proved a theorem regarding faithful matrix representations of almost Abelian Lie groups and explicitly found a faithful finite-dimensional representation when it exists. I also derived an implicit description of the automorphism group of an arbitrary real connected almost Abelian Lie group. The project required three months of preliminary reading from Lie Groups, Lie Algebras, and Representations by Brian C. Hall, as well as an intimate familiarity with several preprints regarding the structure of almost Abelian Lie algebras. I presented my

findings at three different conferences, including the Young Mathematician's Conference at OSU. Though the program is over, I continue to work with the UCSB group remotely, and have nearly proved the general case of the aforementioned theorem. In contrast to my prior experiences with physics, this research was a perfect fit for me and strengthened my resolve to pursue a future in mathematics. We expect to submit a paper this spring.

Do you have Papers/Publications associated with this research project?
No

Do you have Presentations associated with this research project?
Yes

If yes, how many presentations are associated with this work?
3

Citation
Berlow K and Martin IK. Almost Abelian Groups, Their Subgroups and Automorphisms. Poster session presented at: Young Mathematicians Conference; 2019 August 9-11; Columbus, OH.

Campus, Regional, National or International
National

Presentation type
Poster

How are you listed on the presentation?
Presenter

Citation
Martin IK. Almost Abelian Groups, Their Subgroups and Automorphisms. Presentation delivered at: Notre Dame, Summer Workshop in Geometry and Topology; 2019 July 31 - August 02; South Bend, IN.

Campus, Regional, National or International
National

Presentation type
Oral

How are you listed on the presentation?
Presenter

Citation
Berlow K and Martin IK. Almost Abelian Groups, Their Subgroups and Automorphisms. Poster session presented at: UCSB Summer Poster Conference; 2019 August 15. Santa Barbara, CA.

Campus, Regional, National or International
Campus

Presentation type

Poster

How are you listed on the presentation?
Presenter

Additional Research Projects

Research Project #2
Thickness Dependence of Electrical Transport in BSTS Topological Insulators

Starting Month
05

Starting Year
2018

Ongoing
No

Ending Month
09

Ending Year
2018

Average Hours/Week (Academic Year)
4

Average Hours/Week (Summer)
25

Name of Project Mentor
Vikram Deshpande

Title of Project Mentor
Professor

Name of Project Mentor
Su Kong Chong

Title of Project Mentor
Graduate Student

Description of research, including your involvement in AND contribution to the project.
I worked under Su Kong Chong, one of Dr. Deshpande's grad students, to study how the thickness of a Topological Insulator (TI) affects electrical transport. Specifically, we were studying the relationship between the thickness of a bulk-insulating TI and the integer quantum Hall conductance on the TI’s surface. Using polymer film pick-up techniques and a micro manipulator, I designed and built heterostructures (thin, layered materials held together

by van der Waals forces) which enabled us to tune the chemical potential of the TI while it was inside an optical cryostat. These heterostructures were incredibly sensitive, and nearly all my devices eventually failed at some stage of the fabrication process. Although I created several successful devices, when I migrated away from the project to focus on math, the team switched to a different design. Nevertheless, the experience provided me with an understanding of experimental research which will be useful for future collaborations.

Do you have Papers/Publications associated with this research project?
No

Do you have Presentations associated with this research project?
No

Additional Research Projects

Research Project #3
Searching for Dark Photon Dark Matter with Gravitational Wave Detector

Starting Month
09

Starting Year
2018

Ongoing
No

Ending Month
02

Ending Year
2019

Average Hours/Week (Academic Year)
8

Average Hours/Week (Summer)
N/A

Name of Project Mentor
Yue Zhao

Title of Project Mentor
Professor

Description of research, including your involvement in AND contribution to the project.
The dark photon is a theorized extension to the standard model which would serve as a force carrier to a dark-matter analog of the electromagnetic force. Our group was trying to demonstrate that LIGO (Laser Interferometer Gravitational-Wave Observatory) has the

capability to detect dark photon dark matter (DPDM). I helped write the Mathematica code used to generate a mock DPDM signal. I also wrote the analysis code used to convert the data to a frequency series and identify the DPDM signal. I learned to use the Einstein Toolkit, a numerical general relativity simulation package, and helped the cosmology group at Utah install it both locally and on our school's cluster computer. I found the theory fascinating, but the task of implementing it in simulations frustrated me. This realization was the final factor in my decision to pursue a graduate degree in math rather than physics.

Do you have Papers/Publications associated with this research project?
No

Do you have Presentations associated with this research project?
No

Additional Research Projects

Research Project #4
Polynomials Research Class

Starting Month
08

Starting Year
2019

Ongoing
No

Ending Month
12

Ending Year
2019

Average Hours/Week (Academic Year)
12

Average Hours/Week (Summer)
N/A

Name of Project Mentor
Thomas Polstra

Title of Project Mentor
Professor

Description of research, including your involvement in AND contribution to the project.
I researched and presented on Grace's Theorem and the vanishing sets of polynomials over non-algebraically closed fields. This independent project was conducted in a special-topics

research class with the primary objective of exposing students to mathematics rarely addressed in undergraduate coursework. We worked primarily out of the books Polynomials by Prasolov and Algebraic Curves by Fulton. Other topics included insolvability of the quintic, Apolar polynomials, the Gauss-Lucas theorem, the Casas-Alvero conjecture, and the Sendov-Ilieff conjecture. This class was a mix between a supervised reading course and an independent research project. In particular, I learned the power of tackling a problem with approaches from multiple disciplines, for instance, in employing both commutative algebra and algebraic geometry techniques to study vanishing sets in non-algebraically closed fields.

Do you have Papers/Publications associated with this research project?
No

Do you have Presentations associated with this research project?
No

Additional Research Projects

Research Project #5
Novel Detection Techniques for Space Debris in LEO

Starting Month
01

Starting Year
2017

Ongoing
No

Ending Month
04

Ending Year
2017

Average Hours/Week (Academic Year)
6

Average Hours/Week (Summer)
N/A

Name of Project Mentor
Jonathon Barnes

Title of Project Mentor
Professor

Description of research, including your involvement in AND contribution to the project.
I had the opportunity to work with Professor Jonathan Barnes to develop an original project and

conduct research over the length of a semester. I studied the possibility of using reflectivity to detect small particles in Low Earth Orbit. Using survey data collected by NASA, I developed an idealized model of the debris in Low Earth Orbit and found that a cloud of debris with a collective surface area of 7872 m^2 could be feasibly detected over the background of the night sky.

Do you have Papers/Publications associated with this research project?
No

Do you have Presentations associated with this research project?
Yes

If yes, how many presentations are associated with this work?
1

Citation
[redacted] 2017. Space Debris in Low Earth Orbit and How We Might Detect it. SLCC Science Symposium.

Campus, Regional, National or International
Campus

Presentation type
Oral

How are you listed on the presentation?
Presenter

Mentor Recognition Information

Mentor Name
Zhirayr Avetisyan

Mentor Title
Professor

Mentor Name
Henryk Hecht

Mentor Title
Professor

Mentor Name
Adam Boocher

Mentor Title
Professor

Research Skills

Skill Description #1

Problem Dissection – My background has given me a robust approach to problem solving. I am good at isolating pieces of a problem and tackling them using techniques from multiple subfields of math. Physics taught me the value of temporarily lessening rigor to better probe and understand a problem.

Additional Research Skills

Skill Description #2

Software Development – I am proficient in several programming languages including C#, C++, Python, and PHP, and am familiar with software such as Mathematica and Macaulay2. Software is becoming increasingly useful to mathematicians, especially for performing computations and generating examples.

Additional Research Skills

Skill Description #3

Technical Writing and Typesetting – Feedback from professors and mentors over the past three years has helped me become a skilled mathematical writer and an advanced LaTeX user. This is a vital skill, since mathematicians often produce many pages of math every day for a variety of purposes.

Additional Research Skills

Skill Description #4

Literature Familiarity – Through presentations and research projects I have gained a familiarity with the style and language of both commutative algebra and representation theory literature. Fluency in the registers of these fields will be essential for me in grad school and my future career.

Additional Research Skills

Skill Description #5

Device Fabrication – I learned to use precision microscopes and a “micro manipulator” to design and build van der Waals heterostructures for use in an optical cryostat in condensed matter experiments.

Other Activities and Accomplishments

Activity/Accomplishment

BIKES: Commutative Algebra Seminar

Organization (if applicable)

University of Utah

Scope of Activity/Accomplishment

College/University

Role/Involvement
BIKES is a weekly seminar where graduate students take turns delivering 50-minute talks on selected commutative algebra topics. I help organize the weekly meetings, suggest topics, and occasionally deliver the presentation.

Leadership Position
Member

Length of Involvement
Academic Year

Additional Other Activities and Accomplishments

Activity/Accomplishment
TAing and Grading

Organization (if applicable)
University of Utah

Scope of Activity/Accomplishment
College/University

Role/Involvement
I have graded for a variety of classes, most often for PDEs, since freshman year. Last year, I took on the role of an active TA for an intro to real Analysis class, for which I grade assignments and lead discussion sections.

Leadership Position
TA

Length of Involvement
More than one academic year

Additional Other Activities and Accomplishments

Activity/Accomplishment
Math Volunteering and Outreach

Scope of Activity/Accomplishment
Community

Role/Involvement
I am a member of USAC in the math department and participate in miscellaneous math outreach events. For instance, at the request of a local teacher, I visited a 6th grade classroom to talk about "sets and different sizes of infinity."

Leadership Position

Member

Length of Involvement
Academic Year

Additional Other Activities and Accomplishments

Activity/Accomplishment
Tutor and Lab Aide

Organization (if applicable)
University of Utah

Scope of Activity/Accomplishment
College/University

Role/Involvement
For the last two years I have worked as a physics and math tutor in the math center. I also help run our computer lab, where I assist students with computational aspects of their math classes in Python, Matlab, and Maple.

Leadership Position
Tutor, Lab Assistant

Length of Involvement
More than one academic year

Additional Other Activities and Accomplishments

Activity/Accomplishment
Other Seminars

Organization (if applicable)
University of Utah

Scope of Activity/Accomplishment
College/University

Role/Involvement
I attend the weekly representation theory seminars for graduate students. I will begin remotely attending the VaNTAGe SEMINAR with my university's number theory group. I also read "Category Theory in Context" weekly with a group of graduate students

Leadership Position
Member/Organizer

Length of Involvement
Semester

Recognitions

Recognition
Eccles Scholar

Type
College/University

Award Description
This is the most prestigious undergraduate scholarship at the University of Utah and is awarded to 30-people annually. Recipients participate in a selective cohort experience during their time as undergraduate Honors students.

Award Year
2017

Additional Recognitions

Recognition
Award in Astronomy and Astrophysics

Type
College/University

Award Description
The "Paul Gilbert Outstanding Undergraduate Research Award in Astronomy and Astrophysics" is a scholarship awarded to a single undergraduate student with contributions in astronomy or astrophysics research.

Award Year
2019

Additional Recognitions

Recognition
Phi Kappa Phi

Type
National

Award Description
Phi Kappa Phi is a national honors society that recognizes academic excellence. I accepted the invitation to join in 2019.

Award Year
2019

Additional Recognitions

Recognition
Dean's List

Type
College/University

Award Description
I have received the "Dean's List" designation every semester at University of Utah. This is a designation given to students who hold a semester GPA of 3.5 or higher.

Award Year
2017

Current College/University

Institution type:
4-year institution

Field of study
Mathematical Sciences

Mathematical Sciences areas of specialization
Algebra, Number Theory, and Combinatorics

Official cumulative unweighted GPA
3.97

How many credit hours does your school require for graduation?
122

How many credit hours will you achieve as of January 1, 2020?
149

How many credit hours do you plan to achieve for graduation?
203

Expected baccalaureate graduation month
05

Expected baccalaureate graduation year
2021

According to the definition provided above, indicate whether you are a sophomore or junior.
Junior

Matriculation status at the institution you will be attending during the 2020-2021 academic year
Currently Enrolled

Have you been involved in or do you plan to Study Abroad?

No

Coursework

Current Course 1
Representation Theory

Current Course 2
Complex Analysis

Current Course 3
Algebraic Number Theory

Current Course 4
Modern Algebra II

Current Course 5
Clustering (Computer Science)

Current Course 6
Writing in a Research University

Future Course (In Major) 1
Lie Groups and Lie Algebras

Future Course (In Major) 2
Differentiable Manifolds

Future Course (In Major) 3
Complex Manifolds

Future Course (In Major) 4
Algebraic Geometry

Future Course (In Major) 5
Topics in Complex Geometry

Future Course (In Major) 6
Local Cohomology Reading Course

Future Course (Outside Major) 1
Computational Geometry

Future Course (Outside Major) 2
Intro to Gravitation

Future Course (Outside Major) 3
Intro to Particle Physics

Future Course (Outside Major) 4
Solid State Physics II

Future Course (Outside Major) 5
Group Theory (Physics class)

Future Course (Outside Major) 6
Advanced Solid State Physics

Previous Schools Attended

Click SEARCH to select a previous school
230746

School Name
Salt Lake Community College

City
Salt Lake City

State/Territory
UT

Institution type:
2-year institution

Dates attended
August 2017 to Present

Unweighted GPA on a 4.00 scale
3.98

Will you be providing a transcript from this school to your Goldwater Campus Representative?
No

Please explain why you will not be providing a transcript.
I am not providing my transcript because the grades from Salt Lake Community College are already reported on my University of Utah transcript. If there is a reason that a new copy of this transcript is required, then I will gladly provide it.

Future Academic Plans

Is the institution you will be attending for the 2020-2021 academic year the same as your current academic institution?
Yes

Certification and Release

MATRIX REPRESENTATIONS AND AUTOMORPHISM GROUPS OF REAL CONNECTED FINITE-DIMENSIONAL ALMOST ABELIAN LIE GROUPS

In the summer of 2019, I worked with Dr. Avetisyan and a few of his students to study almost Abelian Lie groups (which I will subsequently define). Well studied examples of these Lie groups are prominent in physics and mathematics literature (see [5] and [4]), yet despite their importance, relatively little is known about them in general. My primary role in this research has been to better understand the structure of almost Abelian Lie groups. I successfully classified the central discrete subgroups of simply-connected almost Abelian Lie groups, proved some facts about the automorphism groups of these Lie groups (omitted from this essay), and am currently finishing up a complete classification of those almost Abelian Lie groups which are matrix groups. The success of this project cemented my resolve to become a research mathematician.

It is necessary to understand several definitions prior to the following discussion. A *real Lie group* is a set G which is both a group and a real smooth manifold [3]. A point p on a smooth manifold M be linearly approximated by its tangent space, denoted T_pM. Lie groups have the special property that for all $g \in G$, the tangent space T_gG is canonically isomorphic to the tangent space T_1G at the group identity. This tangent space is called the *Lie Algebra* associated to G, and is often denoted $\mathfrak{g}$. It encodes the local structure of G, and the exponential map $\exp : \mathfrak{g} \to G$ provides a way to move between the two objects. Finally, it is often convenient to relate Lie groups to better understood objects, for instance, matrices. Any closed subgroup of $GL_n(\mathbb{R})$, the group of invertible matrices over real numbers, is itself a Lie group, so in order to understand an arbitrary real Lie group G, one might consider a Lie group homomorphism $\Phi : G \to \mathrm{GL}_n(\mathbb{R})$. Such a homomorphism is called a *matrix representation* of G. If Φ happens to be injective, then we say this representation is *faithful*, and since G is isomorphic to its image $\Phi(G)$ in this case, we say that G is a *matrix Lie group*.

Lie groups and Lie algebras are versatile tools in physics. The Heisenberg group $\mathcal{H}$ and its associated Lie algebra $\mathfrak{h}$, pictured below [3]:

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{R} \right\}, \qquad \mathfrak{h} = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{R} \right\},$$

are among the most infamous and widely applicable. In cosmology, Bianchi algebras of type II-VII appear in homogeneous space time models, and in crystalography, 3 dimensional Lie groups represent the spatial symmetries of anisotropic crystals [1]. These examples all fail to commute in exactly one basis element, a property generalized by the notion of an *almost Abelian Lie algebra*.

An *almost Abelian Lie algebra* is a non-Abelian Lie algebra with a codimension one Abelian subalgebra, and an *almost Abelian Lie group* is a Lie group whose associated Lie algebra is almost Abelian [1]. Well-studied examples of almost Abelian Lie algebras and groups are scattered throughout physics and mathematics literature (see, for instance, [5] and [4]), yet relatively little is known about them in general. Matrix Lie groups, on the other hand, are well understood, so it is natural to ask: which almost Abelian Lie groups admit faithful matrix representations? Zhirayr Avetisyan previously showed that the answer is, "always," for simply-connected Lie groups, but, as I proved, it turns out to be more complicated in the connected case.

It is useful to understand the relationship between connected and simply-connected Lie groups. A *connected* Lie group is one which is generated by the image of its associated Lie algebra under the exponential map [3], or equivalently, is a path-connected Lie group. We say that a Lie group is simply connected if it is contractible, and it is a fact that simply connected Lie groups are unique up to isomorphism. Every connected Lie group G has associated to it a simply connected Lie group $\widetilde{G}$ – called the universal cover of G – which is unique up to isomorphism. The following lemma follows easily from this discussion [3]:

Lemma 1. Let G be a connected matrix Lie group G and $\widetilde{G}$ its universal cover. There exists a discrete normal subgroup $N \subseteq \widetilde{G}$ contained in the center of $\widetilde{G}$ such that $G \cong \widetilde{G}/N$.

The converse of this lemma is also true: every quotient $\widetilde{G}/N$ of the above form is a connected Lie group with universal cover $\widetilde{G}$. In order to classify those real connected almost Abelian Lie groups which admit faithful matrix representations, it therefore suffices to determine which quotient groups $\widetilde{G}/N$ admit faithful matrix representations, where $\widetilde{G}$ is a simply connected almost Abelian Lie group and N is as above.

An arbitrary $n+1$ dimensional simply connected almost Abelian Lie group G admits the following faithful matrix representation [2]:

$$\varphi : G \to \mathrm{GL}_{n+2}(\mathbb{R}), \quad (v,t) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ v & e^{tJ(\aleph)} & 0 \\ 0 & 0 & e^t \end{pmatrix} \in \mathrm{End}(\mathbb{R}^{n+2}), \quad \forall (v,t) \in \mathbb{R}^n \oplus_{\mathbb{R}} \mathbb{R} \tag{1}$$

where $J(\aleph)$ is a $n \times n$ Jordan matrix determined by the multiplicity function $\aleph$. Using this representation, I computed the center $Z(G)$:

$$Z(G) \simeq \left\{ \begin{pmatrix} 1 & 0 & 0 \\ v & \mathbb{1} & 0 \\ 0 & 0 & e^t \end{pmatrix} \;\middle|\; J(\aleph)v = 0,\ \ e^{tJ(\aleph)} = \mathbb{1} \right\}. \tag{2}$$

It is a fact that if $\mathfrak{g}$ is the Lie algebra associated to G then the image $\exp(Z(\mathfrak{g}))$ generates the identity component of $Z(G)$ [3]. In this case however, I noticed that the restriction $\exp : \mathfrak{g} \to G$ to $Z(\mathfrak{g}) \subset \mathfrak{g}$ is in fact a diffeomorphism, i.e. the identity component of $Z(G)$ is *exponential*.

Since this restriction of $\exp$ is a diffeomorphism, we may consider its inverse $\exp^{-1}$. Letting d denote $\dim(Z(\mathfrak{g}))$, $\exp^{-1}$ can be extended to a map

$$\alpha : Z(G) \to Z(\mathfrak{g}) \oplus \mathbb{R} \cong \mathbb{R}^{d+1}, \quad (v,t) \mapsto (v,t) \in \mathbb{R}^{d+1}$$

where $Z(\mathfrak{g}) \oplus \mathbb{R} \cong \mathbb{R}^{d+1}$ is an isomorphism of Abelian Lie algebras. Thus, α is an embedding of $Z(G)$ into $\mathbb{R}^{d+1}$, and it is easily shown to be a homomorphism of groups as well as a continuous, open map. It follows that a subgroup $N \subset Z(G)$ is discrete if and only if $\alpha(N) \subset \mathbb{R}^{d+1}$ is a discrete additive subgroup, giving us a correspondence between the discrete central subgroups of G and the discrete additive subgroups of $\mathbb{R}^{d+1}$ contained in $\alpha(Z(G))$. The discrete additive subgroups of $\mathbb{R}^{d+1}$ are well known – they are free groups of rank at most $d+1$ – giving us the following proposition:

Proposition 1. Every discrete, central subgroup $N \subset G$ of a real, finite-dimensional simply connected almost Abelian Lie group G is a free group with rank at most $d+1$ and is generated by set S whose image under the embedding $\alpha : Z(G) \hookrightarrow R^{d+1}$ is linearly independent over $\mathbb{R}$.

Having proven a complete classification of discrete central subgroups, I began to study the structure of arbitrary quotients G/N. This proved difficult to do in general. Every almost Abelian Lie algebra takes the form $\mathfrak{g} = L_0 \oplus_{\mathbb{R}} W$, where L_0 is an indecomposable almost Abelian Lie subalgebra of $\mathfrak{g}$ and W is an Abelian subalgebra of $\mathfrak{g}$ [1], and L_0 often contains a copy of the Heisenberg algebra. It is worth noting that in this case $G = \exp(L_0) \times \exp(W) \times T$, where $T = \{t \in \mathbb{R} \mid e^t J = 0\}$.

Famously, quotients of the Heisenberg group do not admit faithful matrix representations [3], so in the case that the intersection $N \cap \exp(L_0)$ is nontrivial, I proved G/N necessarily contains a subgroup isomorphic to a quotient of the Heisenberg group and is consequently not a matrix group. On the other hand, when N is entirely contained in $\exp(W) \times T$, the quotient G/N *does* in fact admit a faithful matrix representation. Because N has a $\mathbb{R}$-linearly independent generating set, we may construct a map $\Phi : G \to \mathrm{GL}_{n+2}(\mathbb{R})$ with kernel N. This induces an isomorphism $G/N \cong \Phi(G)$, and shows that G/N is a matrix Lie group.

Of course, even if N is not contained in $\exp(W) \times T$, it may have trivial intersection with $\exp(L_0)$. For example, consider the following simply-connected almost Abelian Lie group G and it's discrete central subgroup N:

$$G = H \times \mathbb{R} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & t & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z, t \in \mathbb{R} \right\}, \qquad N = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ m & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m & 0 & 0 & 1 \end{pmatrix} \middle| \; m \in \mathbb{Z} \right\}.$$

Here, N has trivial intersection with $\exp(L_0)$ but is not contained in $\exp(W) \times T$. However, under the map $\sigma \in \mathrm{Aut}(G)$ defined $\sigma(x, y, z, t) = (x - z, y, z, t)$, we have that $\sigma(N) \subset \exp(W) \times T$, which means $G/\sigma(N)$ is a matrix group. Since σ is an automorphism, $G/N \cong G/\sigma(N)$, so G/N is a matrix group too. This example suggests, in the case $N \cap \exp(L_0) = \{1\}$, that G/N is a matrix Lie group. A proof of this result would finalize the classification of those real connected almost Abelian Lie groups which admit faithful matrix representations.

This final case proved too elusive to tackle in a summer. Nevertheless, I was accepted to present this research at two conferences, one at the summer workshop for geometry and topology at Notre Dame and another at the Young Mathematicians Conference hosted by OSU. Though I have left Santa Barbara, I continue to work with Zhirayr Avetisyan et al. remotely. We have compiled our findings into a preprint which will soon be available on arXiv, and hope to submit our work to academic journals in the future.

References

[1] Zhirayr Avetisyan. Structure of almost abelian lie algebras. *ArXiv*, 2016.

[2] Zhirayr Avetisyan. Jordanable almost abelian lie algebras. *ArXiv*, 2018.

[3] Brian Hall. *Lie Groups, Lie Algebras, and Representations*. Springer, 2nd edition, 2003.

[4] G. P. Parry and Rainer Verch. Discrete structures in continuum descriptions of defective crystals. *Classical and Quantum Gravity*, Aug 2013.

[5] Sundaram Thangavelu. *Harmonic Analysis on the Heisenberg Group (Progress in Mathematics)*. Berkhäuser, 1st edition, 1998.